OPEN ACCESS
IOP Publishing | ISECAEP
Journal of Reliability Science and Engineering
J. Reliab. Sci. Eng. **2** (2026) 035002 (16pp)
https://doi.org/10.1088/3050-2454/ae8dd2

Reliability allocation of multicomponent systems based on non-homogeneous pure birth process under dynamic environment

Zhou Fang[1], **Wenjie Dong**[1,*] **and Yingsai Cao**[2,*]

[1] College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 211106, People's Republic of China
[2] School of Management, Jiangsu University, Zhenjiang, Jiangsu 212013, People's Republic of China

E-mail: dongwenjie@nuaa.edu.cn, caoyingsai@foxmail.com and zhoufang1@nuaa.edu.cn

Received 19 May 2026, revised 1 July 2026
Accepted for publication 20 July 2026
Published 5 August 2026

Abstract

Multicomponent systems operating in dynamic environments are affected by cumulative environmental shocks and dependent component lifetimes, which may lead to inaccurate reliability evaluation and inefficient reliability allocation if ignored. To address this issue, this paper develops a non-homogeneous pure birth process (NHPBP)-driven stochastic time-scale model for multicomponent systems. The environmental shock-counting process is modeled by an NHPBP, while the stochastic time scale is constructed as a path-dependent cumulative exposure functional driven by the whole shock history. Based on the Laplace transform of the stochastic time scale, component survival functions, joint survival functions, and reliability expressions for typical system structures are derived. Positive quadrant dependence among component lifetimes is also established. Furthermore, a cost-oriented reliability allocation model is formulated under a system-level reliability constraint and solved using a genetic algorithm. Numerical experiments on an automotive braking system show that the conventional non-homogeneous Poisson process (NHPP)-based model may overestimate system reliability under cumulative shock amplification, whereas the proposed NHPBP-based allocation model provides feasible allocation schemes under dynamic environments.

Keywords: non-homogeneous pure birth process (NHPBP), reliability allocation, system reliability, genetic algorithm (GA)

* Authors to whom any correspondence should be addressed.

Original content from this work may be used under the terms of the Creative Commons Attribution 4.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

© 2026 The Author(s). Published by IOP Publishing Ltd on behalf of the Institute of Systems Engineering, China Academy of Engineering Physics

1. Introduction

In modern engineering, multicomponent systems are widely used in complex applications such as automobile assembly lines, aerospace equipment, and power grids. Their reliability depends not only on individual component quality, but also on inter-component collaboration and adaptability to complex operating environments [1]. In practice, such systems are often exposed to time-varying operating conditions and shock disturbances [2], which may accelerate component degradation and introduce additional uncertainty into the failure process. For example, aero-engine components operate under extreme thermal and mechanical stresses, and their performance states may evolve dynamically under coupled operating conditions [3]. Similar dynamic effects are observed in load-sharing systems, where component failures redistribute operational burdens to surviving components. Jia *et al* [4] proposed an extended Markov process method for dynamic load-sharing systems. However, many existing load-sharing models do not explicitly incorporate cumulative shock amplification caused by dependent environmental increments. Therefore, dynamic operating conditions and cumulative environmental shocks should be considered in multicomponent system reliability evaluation.

Meanwhile, components in multicomponent systems are usually not independent, because shared dynamic environments and coupled operating mechanisms may induce dependent component lifetimes [5]. The degradation or failure of one component often indicates that the system has experienced more severe environmental shocks, which may also affect other components [6]. In addition, component failure may change load distribution, energy transmission paths, and operating states, thereby influencing the degradation of other components. Ignoring such dependence may impair reliability evaluation accuracy [7]. Existing studies have considered failure-rate interactions and condition-based maintenance decisions for multicomponent systems under imperfect information [8, 9]. For example, front and rear brake pads are exposed to common driving patterns, road conditions, and braking intensity, which may result in statistically dependent wear trajectories and correlated lifetimes [10]. Therefore, component dependence under shared dynamic environments should be explicitly considered in reliability modeling and allocation.

Recent studies on maintenance and risk-control decisions further show that degradation uncertainty, condition monitoring information, mission termination or abort decisions, and technological renewal effects can significantly influence system-level reliability management [11–13]. These studies provide useful decision-making perspectives for dynamic reliability management, but they do not directly address reliability allocation under dependent component lifetimes induced by a common stochastic time scale. This motivates the development of a reliability allocation framework that explicitly connects dynamic environmental shocks, component lifetime dependence, and system-level reliability constraints.

Traditional reliability models often assume a time-invariant operating environment or independent component failures, which is inadequate for systems operating under cumulative and stochastic environmental shocks. Although multivariate lifetime models and Copula methods [14, 15] provide useful tools for describing statistical dependence, they may suffer from limited physical interpretability and insufficient flexibility in representing dynamic aging mechanisms [16]. Based on the classical cumulative exposure model [17], Hong *et al* [10] introduced a stochastic time-scale model, in which the calendar time t is transformed into an equivalent operating time $H(t)$ to reflect cumulative environmental exposure. Conditional on the stochastic time scale $H(t)$, the survival function of the ith component is expressed as

$$P\{T_i > t \mid H(t)\} = \exp\left[-\eta_i H(t)\right], \tag{1}$$

where $\eta_i (\eta_i > 0)$ denotes the baseline failure rate of the ith component. In this formulation, the unconditional survival function is obtained by $P(T_i > t) = E\{\exp[-\eta_i H(t)]\}$. When $H(t)$ is deterministic, common lifetime distributions can be obtained by specifying different time-scale functions; for example, $H(t) = t^{\alpha}$ leads to the Weibull distribution [18], $H(t) = 1 - \exp(-\beta t)$ leads to an exponential time-scale transformation [19], and $H(t) = \log(1 + t/\beta)$ leads to the Lomax distribution. When the system operates in a dynamically changing environment, $H(t)$ should be modeled as a stochastic process. Existing studies have used monotone processes such as compound Poisson processes, Gamma processes, and inverse Gaussian processes [10], exponential dispersion processes [20], and Markov chains [21] to characterize stochastic environmental effects. However, these models mainly rely on independent-increment or externally specified environmental processes, which may be insufficient when historical shocks increase both future shock occurrence rate and component aging exposure. To address this issue, Liu and Wang [16] modeled $H(t)$ using the generalized Pólya process (GPP), a dependent-increment process. In this paper, the environmental shock process $N(t)$ is modeled by an non-homogeneous pure birth process (NHPBP), while the stochastic time scale $H(t)$ is constructed as a path-dependent cumulative exposure functional driven by the whole shock history, i.e. $H(t) = \int_0^t b_{N(s)}\, \mathrm{d}\Lambda(s)$.

Copula-based and multivariate lifetime models provide statistical descriptions of component dependence. The dependence structure is usually specified at the distribution level, and the physical evolution of environmental shocks is not modeled directly. non-homogeneous Poisson process (NHPP)-based models can represent time-varying shock arrivals. Their shock intensity is governed by calendar time, and accumulated shocks do not change the occurrence tendency of future shocks. GPP-based stochastic time-scale models introduce reinforcement into environmental exposure. The separation between the shock-counting mechanism and the aging exposure mechanism is still less explicit for reliability allocation under dependent component lifetimes.

The NHPBP-driven formulation used in this paper is motivated by two physical mechanisms in dynamic environments. The first mechanism is state-dependent shock amplification. After more shocks have accumulated, the operating

Table 1. Comparison of representative modeling approaches.

Approach	Shock modeling	Aging exposure
NHPP	Time-dependent	External or deterministic
Gamma process	Independent increments	Monotone degradation
GPP	Reinforced process	Stochastic time scale
Copula	Not process-based	Not explicitly modeled
Proposed model	State-dependent NHPBP	Path-dependent $H(t)$

environment may become more unstable, and future shocks may occur more frequently. The second mechanism is path-dependent cumulative aging. Two shock histories with the same final shock count may generate different aging exposure when the shocks occur at different times. In the proposed framework, the environmental shock-counting process $N(t)$ is modeled by an NHPBP, and the cumulative aging exposure is represented through a stochastic time scale that accumulates environmental effects over the shock history. This formulation separates shock occurrence from aging exposure and provides a process-based description of dependent component lifetimes under cumulative environmental shocks.

To further clarify the modeling position of the proposed framework, representative stochastic time-scale and dependence modeling approaches are compared in table 1.

As shown in table 1, each approach has its own modeling scope. NHPP-based models describe time-varying shock arrivals, with no state amplification. Gamma-process-based models are suitable for monotone degradation modeling, with limited description of shock states. GPP-based models capture reinforcement effects through a stochastic time scale, with a less explicit separation between shock occurrence and aging exposure. Copula-based methods describe statistical dependence, with no direct process-based representation of shock accumulation. The proposed model explicitly distinguishes $N(t)$ from $H(t)$, making it possible to describe state-dependent shock occurrence and path-dependent aging exposure within the same reliability allocation framework.

Reliability allocation is a critical step in system reliability design, aiming to decompose system-level reliability requirements into component-level reliability targets while balancing reliability improvement and economic cost. Traditional allocation methods often assume mutually independent component failures [22]. However, in dynamic environments, component lifetimes may be dependent due to shared environmental exposure, performance coupling, and cumulative shock effects [2]. Several studies have investigated reliability analysis and allocation under dependence, parameter correlations, or complex engineering constraints [23, 24]. In addition, maintenance and quality-inspection decisions have also been integrated in reliability-oriented optimization for manufacturing networks [25]. Nevertheless, these methods are not directly integrated with a stochastic time scale that reflects cumulative environmental shocks and path-dependent aging exposure. Therefore, this paper formulates a reliability allocation model based on the proposed NHPBP-driven stochastic time scale. Since the resulting optimization problem is nonlinear and constrained, GA-based methods are adopted to obtain component-level reliability allocation schemes under system-level reliability constraints [24, 26].

The main contributions of this paper are summarized as follows:

1. An NHPBP-driven stochastic time-scale model is developed to describe state-dependent shock amplification and path-dependent cumulative aging.
2. A reliability analysis method is established for multicomponent systems with dependent component lifetimes under the common stochastic time scale.
3. A cost-oriented reliability allocation model is formulated under dependent component lifetimes and cumulative shock amplification.

The rest of this article is organized as follows. Section 2 introduces the proposed NHPBP-driven stochastic time-scale model. Section 3 analyzes component dependence. Section 4 derives system reliability expressions. Section 5 formulates the reliability allocation model and GA-based solution procedure. Section 6 presents numerical results. Finally, section 7 concludes this article.

2. Non-homogeneous pure birth process

The NHPBP is a counting process proposed in [27], and it has been widely used in maintenance modeling and reliability analysis. Traditional shock-based maintenance models usually describe system failures through external shock arrivals. Esary and Marshall [28] assumed that shock arrivals follow a homogeneous Poisson process, while A-Hameed and Proschan [29] considered an NHPP. However, the NHPP only depends on the system age and cannot describe the cumulative influence of historical shocks on future shock occurrences. To incorporate such state dependence, A-Hameed and Proschan [30] proposed a pure birth shock model based on a nonstationary Markov process, in which, given that k shocks have occurred in $[0,t]$, the probability of one shock occurring in $(t,t+\Delta]$ is $\lambda_k\lambda(t)\Delta + o(\Delta)$. Based on [30], Sheu *et al* [27] proposed an NHPBP repair model and studied the corresponding age replacement strategy. Sheu *et al* [31] further investigated a block replacement strategy, while Chien [32] derived the optimal preventive maintenance policy for NHPBP repairable systems with a free-repair warranty. Sheu *et al* [33] studied the minimum replacement policy to minimize the expected maintenance cost rate, and Chien *et al* [34] extended the model to periodic preventive maintenance and derived the optimal (N^*, T^*). Beyond maintenance optimization, NHPBP has also been applied in reliability research. Shaked *et al* [35] characterized I^{∞}-spherical density using NHPBP epoch times and provided its properties. Existing studies mainly apply NHPBP to single-system maintenance and replacement optimization, whereas its application to multicomponent system reliability remains limited. Therefore, this paper introduces NHPBP into multicomponent reliability analysis to describe cumulative environmental shocks and the induced dependence among

component lifetimes. Next, the definition of the NHPBP is presented.

Definition 1: If a counting process $\{N(t), t \geqslant 0\}$ is a non-homogeneous continuous time Markov process with the following conditions:

1. $N(0) = 0$
2. $P\{N(t+h) - N(t) = 1 \mid N(t) = k\} = \lambda_k(t)h + o(h)$
3. $P\{N(t+h) - N(t) \geqslant 2 \mid N(t) = k\} = o(h)$

Then the process is called an NHPBP with the state-dependent intensity function $\{\lambda_k(t), k = 0, 1, 2, \cdots\}$.

From Definition 1, if $\lambda_k(t) = \lambda(t)$ for all $k \leqslant 0$, the NHPBP reduces to an NHPP. Therefore, compared with the NHPP, the main feature of the NHPBP is that the shock occurrence intensity depends not only on calendar time, but also on the current shock state. This property enables the process to describe the cumulative influence of historical shocks on future shock occurrences.

In this paper, the NHPBP is used to model the environmental shock-counting process $N(t)$. To characterize the cumulative amplification effect of historical shocks while avoiding explosive growth of the birth rates, the state-dependent intensity is specified as

$$\lambda_k(t) = a_k \lambda(t), \tag{2}$$

where

$$a_k = 1 + \omega k. \tag{3}$$

Here, $\lambda(t)$ denotes the baseline shock intensity, and a_k is the state-dependent amplification factor. The parameter $\omega \geqslant 0$ measures the effect of accumulated shocks on future shock occurrence rate. When $\omega = 0$, $a_k = 1$, and the model reduces to an NHPP; when $\omega > 0$, historical shocks increase the subsequent shock intensity.

From an engineering perspective, ω reflects the sensitivity of future shock occurrence to the accumulated shock history. A larger ω means that, after more shocks have occurred, the system is more likely to experience subsequent shocks. For example, repeated severe braking, vibration disturbances, or thermal-mechanical load cycles may make the operating environment more unstable and increase the occurrence tendency of future shock events.

The baseline intensity $\lambda(t)$ describes the time-varying tendency of environmental shocks before state-dependent amplification is applied. In this paper, it is assumed to follow a Weibull-type intensity function:

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}, \tag{4}$$

with cumulative baseline intensity

$$\Lambda(t) = \int_0^t \lambda(x)\,\mathrm{d}x = \left(\frac{t}{\theta}\right)^{\beta}. \tag{5}$$

The parameter $\beta > 0$ controls the time-varying trend of the baseline shock intensity, and $\theta > 0$ is the scale parameter. Thus, $\lambda(t)$ captures the calendar-time effect, while a_k captures the shock-history effect.

Let

$$p_k(t) = P\{N(t) = k\}, \quad k = 0, 1, 2, \cdots. \tag{6}$$

For the NHPBP defined above, the state probabilities satisfy the following Kolmogorov forward equations

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -a_0\lambda(t)p_0(t), \tag{7}$$

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = a_{k-1}\lambda(t)p_{k-1}(t) - a_k\lambda(t)p_k(t), \tag{8}$$

with the initial conditions

$$p_0(0) = 1, \quad p_k(0) = 0. \tag{9}$$

These equations describe the probability flow from state $k-1$ to state $k (k \geqslant 1)$. In particular, since $a_0 = 1$, the probability of no shock occurring in $[0, t]$ is

$$p_0(t) = \exp[-\Lambda(t)]. \tag{10}$$

The recursive solution of $p_k(t)$ and the non-explosiveness of the linear amplification form are provided in appendix A.

The counting process $N(t)$ describes the stochastic evolution of environmental shocks. Based on this process, the next section constructs a path-dependent stochastic time scale and analyzes the dependence induced among component lifetimes.

3. Component lifetime model and dependence

In this section, the NHPBP shock-counting process $N(t)$ defined in section 2 is used to construct a stochastic time scale for component lifetime modeling. Since all components operate under the same dynamic environment, they share the same shock history. The resulting common stochastic exposure induces dependence among component lifetimes in the unconditional sense.

The stochastic time scale is defined as a cumulative exposure functional driven by the whole path of $N(t)$:

$$H(t) = \int_0^t b_{N(s)}\,\mathrm{d}\Lambda(s), \tag{11}$$

where

$$b_k = 1 + \gamma k. \tag{12}$$

b_k is the aging-rate amplification factor when k shocks have occurred, and $\gamma \geqslant 0$ measures the influence of accumulated shocks on the component aging rate. When $\gamma = 0$, the accumulated shocks do not further amplify the aging exposure and $H(t)$ reduces to $\Lambda(t)$. When $\gamma > 0$, a higher shock state leads to a larger aging-rate amplification factor. In engineering systems, this may correspond to accelerated wear, fatigue accumulation, or thermal damage caused by repeated shocks.

If the shock epochs in $[0,t]$ are $0<\tau_1<\cdots<\tau_n\leqslant t$, with $\tau_0=0$ and $\tau_{n+1}=t$, then

$$H(t)=\sum_{j=0}^{n}(1+\gamma j)\left[\Lambda(\tau_{j+1})-\Lambda(\tau_j)\right]. \tag{13}$$

Therefore, $H(t)$ depends not only on the final shock count $N(t)$, but also on the occurrence times of shocks. This definition preserves both the elapsed calendar time and the accumulated shock history.

Conditional on $H(t)$, the lifetime of the ith component is assumed to satisfy

$$P\{T_i>t\mid H(t)\}=\exp\left[-\eta_i H(t)\right], \tag{14}$$

where T_i is the lifetime of component i, and $\eta_i>0$ is its baseline failure rate. This assumption is equivalent to specifying the cumulative hazard of component i as $\eta_i H(t)$ under the realized environmental exposure.

In addition, conditional on the common stochastic time scale, the residual failure mechanisms of different components are assumed to be independent. Therefore, the dependence among component lifetimes is induced by the shared random environment, rather than by conditional residual failure interactions.

Proposition 1: *For the NHPBP-driven stochastic time scale defined in equation (11), the survival function of component i is*

$$S_i(t)=P\{T_i>t\}=L_H(\eta_i;t), \tag{15}$$

where

$$L_H(q;t)=E\{\exp\left[-qH(t)\right]\}\,(q\geqslant 0), \tag{16}$$

is the Laplace transform of the stochastic time scale $H(t)$.

Proof: Taking expectation with respect to $H(t)$, we obtain

$$\begin{aligned}S_i(t)&=E\{P\{T_i>t\mid H(t)\}\}=E\{\exp\left[-\eta_i H(t)\right]\}\\&=L_H(\eta_i;t).\end{aligned} \tag{17}$$

This completes the proof.

To evaluate $L_H(q;t)$, define the weighted state probability

$$m_k(t;q)=E\left[\exp\left[-qH(t)\right]\mathbf{1}_{\{N(t)=k\}}\right]. \tag{18}$$

Then

$$L_H(q;t)=\sum_{k=0}^{\infty}m_k(t;q). \tag{19}$$

For the NHPBP with $\lambda_k(t)=a_k\lambda(t)$ and the stochastic time scale in equation (11), the weighted state probabilities satisfy

$$\frac{\mathrm{d}m_0(t;q)}{\mathrm{d}t}=-(a_0+qb_0)\lambda(t)m_0(t;q), \tag{20}$$

$$\frac{\mathrm{d}m_k(t;q)}{\mathrm{d}t}=a_{k-1}\lambda(t)m_{k-1}(t;q)-(a_k+qb_k)\lambda(t)m_k(t;q), \tag{21}$$

with initial conditions $m_0(0;q)=1, m_k(0;q)=0(k\geqslant 1)$.

The derivation of the weighted state equations is provided in appendix B.

After the time transformation $u=\Lambda(t)$, denote the weighted state vector by

$$\mathbf{m}(u;q)=(m_0(u;q),m_1(u;q),m_2(u;q),\ldots)^T,$$

with

$$\mathbf{e}_0=(1,0,0,\ldots)^T,\quad \mathbf{1}=(1,1,1,\ldots)^T.$$

Then the weighted state equations can be written as

$$\frac{\mathrm{d}\mathbf{m}(u;q)}{\mathrm{d}u}=B(q)\mathbf{m}(u;q), \tag{22}$$

where $B(q)$ is a lower bidiagonal matrix with nonzero elements

$$B(q)_{k,k}=-(a_k+qb_k),k=0,1,2,\cdots, \tag{23}$$

$$B(q)_{k,k-1}=a_{k-1},k\geqslant 1. \tag{24}$$

Here, $a_k=1+\omega k$ and $b_k=1+\gamma k$. Hence,

$$L_H(q;t)=\mathbf{1}^T\exp\left[B(q)\Lambda(t)\right]\mathbf{e}_0. \tag{25}$$

In numerical implementation, the infinite-dimensional matrix $B(q)$ is truncated to a finite dimension. The truncation level is selected according to the retained probability mass of the shock-counting process under $q=0$. A sufficiently large truncation dimension is used so that the omitted tail probability is negligible over the specified mission interval. This criterion ensures that the matrix-exponential approximation preserves the main probability mass of the NHPBP state distribution.

Proposition 2: *For two different components s and m, their joint survival function at a common mission time t is*

$$S_{s,m}(t,t)=P\{T_s>t,T_m>t\}=L_H(\eta_s+\eta_m;t). \tag{26}$$

Proof: By the conditional independence assumption,

$$\begin{aligned}P\{T_s>t,T_m>t\mid H(t)\}&=P\{T_s>t\mid H(t)\}P\{T_m>t\mid H(t)\}\\&=\exp\left[-\eta_s H(t)\right]\exp\left[-\eta_m H(t)\right]\\&=\exp\left[-(\eta_s+\eta_m)H(t)\right].\end{aligned}$$

Taking expectation with respect to $H(t)$, we have

$$S_{s,m}(t,t)=E\{\exp\left[-(\eta_s+\eta_m)H(t)\right]\}=L_H(\eta_s+\eta_m;t).$$

This completes the proof.

The joint survival function for unequal mission times is given in appendix C.

Proposition 3: *Under the common stochastic time scale, the lifetimes of components s and m are positively quadrant dependent, i.e.*

$$P\{T_s>t_s,T_m>t_m\}\geqslant P\{T_s>t_s\}P\{T_m>t_m\}, \tag{27}$$

for any $t_s,t_m\geqslant 0$.

Proof: See appendix C.

Proposition 3 shows that the shared stochastic time scale induces positive dependence among component lifetimes. Therefore, ignoring such dependence may lead to biased system reliability evaluation and suboptimal reliability allocation. Based on the above component reliability and dependence results, the next section derives the reliability functions of typical multicomponent systems.

4. Reliability analysis

Based on the component lifetime model in section 3, this section derives the reliability expressions of typical multicomponent systems under the NHPBP-driven stochastic time scale. All components share the same stochastic time scale $H(t)$, which is generated by the common shock history $N(t)$. Conditional on $H(t)$, the residual failure mechanisms of different components are treated as independent. After averaging over $H(t)$, component lifetimes become dependent in the unconditional sense. This dependence enters the reliability expressions of series, parallel, series-parallel, and parallel-series systems through the joint survival probabilities expressed by $L_H(q;t)$. System reliability is evaluated through the common exposure $H(t)$, rather than by directly multiplying marginal component reliabilities.

Consider a system consisting of K components with lifetimes $T_1, T_2, \cdots, T_K$. Let

$$X_i(t) = \mathbf{1}_{\{T_i > t\}},$$

where $X_i(t) = 1$ indicates that component i works at time t, and $X_i(t) = 0$ otherwise.

Denote the component state vector by

$$\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_K(t)).$$

Let $\phi(\mathbf{x})$ be the structure function of the system:

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system works under state } \mathbf{x}, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Then the system reliability at mission time t is

$$R(t;\boldsymbol{\eta}) = E[\phi(\mathbf{X}(t))] = P\{\phi(\mathbf{X}(t)) = 1\}, \tag{29}$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_K)^T$ is the vector of component baseline failure rates.

For any subset $\mathcal{A} \subseteq \{1, 2, \cdots, K\}$, define

$$\eta_{\mathcal{A}} = \sum_{i \in \mathcal{A}} \eta_i, \tag{30}$$

with the convention $\eta_{\emptyset} = 0$. For any nonempty subset $\mathcal{A}$, the joint survival probability of the components in $\mathcal{A}$ is

$$P\left(\bigcap_{i \in \mathcal{A}} \{T_i > t\}\right) = E\{\exp[-\eta_{\mathcal{A}} H(t)]\} = L_H(\eta_{\mathcal{A}}; t). \tag{31}$$

Equation (31) is the basic building block for evaluating system reliability under the common stochastic time scale.

For a general coherent system, let

$$\mathcal{W}(\mathbf{x}) = \{i : x_i = 1\}, \qquad \mathcal{F}(\mathbf{x}) = \{i : x_i = 0\}.$$

By expanding the conditional state probability with respect to the common $H(t)$, we have

$$\begin{aligned} P\{\mathbf{X}(t) = \mathbf{x}\} &= E\left[\prod_{i \in \mathcal{W}(\mathbf{x})} \mathrm{e}^{-\eta_i H(t)} \prod_{i \in \mathcal{F}(\mathbf{x})} \left(1 - \mathrm{e}^{-\eta_i H(t)}\right)\right] \\ &= \sum_{\mathcal{B} \subseteq \mathcal{F}(\mathbf{x})} (-1)^{|\mathcal{B}|} L_H\left(\eta_{\mathcal{W}(\mathbf{x}) \cup \mathcal{B}}; t\right). \end{aligned} \tag{32}$$

Substituting equation (32) into equation (29) gives a unified reliability expression. For common system structures, more compact forms are obtained below.

4.1. Series system

For a series system with K components, the system works only when all components work, and its lifetime is

$$T_{\mathrm{s}} = \min\{T_1, T_2, \cdots, T_K\}.$$

Therefore,

$$R_{\mathrm{s}}(t;\boldsymbol{\eta}) = P\left(\bigcap_{i=1}^{K} \{T_i > t\}\right) = L_H\left(\sum_{i=1}^{K} \eta_i; t\right). \tag{33}$$

Thus, the series system reliability is evaluated at the aggregate baseline failure rate of all components.

4.2. Parallel system

For a parallel system with K components, the system works if at least one component works, and its lifetime is

$$T_{\mathrm{p}} = \max\{T_1, T_2, \cdots, T_K\}.$$

Let

$$\mathcal{K} = \{1, 2, \cdots, K\}.$$

Using the inclusion-exclusion principle, the reliability of the parallel system is

$$\begin{aligned} R_{\mathrm{p}}(t;\boldsymbol{\eta}) &= P\left(\bigcup_{i \in \mathcal{K}} \{T_i > t\}\right) \\ &= \sum_{\emptyset \neq \mathcal{A} \subseteq \mathcal{K}} (-1)^{|\mathcal{A}|+1} P\left(\bigcap_{i \in \mathcal{A}} \{T_i > t\}\right) \\ &= \sum_{\emptyset \neq \mathcal{A} \subseteq \mathcal{K}} (-1)^{|\mathcal{A}|+1} L_H(\eta_{\mathcal{A}}; t). \end{aligned} \tag{34}$$

For the special case of two parallel components, equation (34) reduces to

$$R_{\mathrm{p}}(t) = L_H(\eta_1; t) + L_H(\eta_2; t) - L_H(\eta_1 + \eta_2; t). \tag{35}$$

This expression differs from the independent-failure formula, because the joint survival term is evaluated through the common stochastic time scale.

4.3. Series-parallel system

Consider a series-parallel system consisting of P subsystems connected in series. The rth subsystem contains n_r components connected in parallel, where $r = 1, 2, \cdots, P$. Let T_{rj} and η_{rj} denote the lifetime and baseline failure rate of the jth component in the rth subsystem, respectively. Define

$$\mathcal{C}_r = \{(r,1), (r,2), \cdots, (r,n_r)\}.$$

For any subset $\mathcal{A}_r \subseteq \mathcal{C}_r$, define

$$\eta_{\mathcal{A}_r} = \sum_{(r,j)\in\mathcal{A}_r} \eta_{rj}.$$

The rth parallel subsystem works if at least one of its components works. Since the subsystems are connected in series, the whole system works only when every subsystem works. Conditional on $H(t)$, the probability that the rth parallel subsystem works is

$$1 - \prod_{j=1}^{n_r} \left(1 - \mathrm{e}^{-\eta_{rj} H(t)}\right).$$

Thus, the reliability of the series-parallel system is

$$\begin{aligned} R_{\mathrm{sp}}(t;\boldsymbol{\eta}) &= E\left\{\prod_{r=1}^{P}\left[1 - \prod_{j=1}^{n_r}\left(1 - \mathrm{e}^{-\eta_{rj}H(t)}\right)\right]\right\} \\ &= \sum_{\substack{\emptyset\neq\mathcal{A}_r\subseteq\mathcal{C}_r \\ r=1,\cdots,P}} (-1)^{\sum_{r=1}^{P}|\mathcal{A}_r|+P} L_H\left(\sum_{r=1}^{P}\eta_{\mathcal{A}_r}; t\right). \end{aligned} \tag{36}$$

Equation (36) expresses the reliability of a series-parallel system as a finite sum of Laplace transforms with different aggregate baseline failure rates.

4.4. Parallel-series system

Consider a parallel-series system consisting of M subsystems connected in parallel. The rth subsystem contains n_r components connected in series, where $r = 1, 2, \cdots, M$. Let T_{rj} and η_{rj} denote the lifetime and baseline failure rate of the jth component in the rth subsystem, respectively. The rth series subsystem works only when all its components work.

Therefore, its reliability is

$$R_r(t) = P\left(\bigcap_{j=1}^{n_r}\{T_{rj} > t\}\right) = L_H\left(\sum_{j=1}^{n_r}\eta_{rj}; t\right). \tag{37}$$

The whole system works if at least one series subsystem works. Because different subsystems share the same stochastic time scale, the subsystem events are not independent in the unconditional sense.

Let $\mathcal{M} = \{1, 2, \cdots, M\}$, for any nonempty subset $\mathcal{D} \subseteq \mathcal{M}$, define

$$\eta_{\mathcal{D}}^{\mathrm{ps}} = \sum_{r\in\mathcal{D}}\sum_{j=1}^{n_r}\eta_{rj}. \tag{38}$$

Using the inclusion-exclusion principle, the reliability of the parallel-series system is

$$\begin{aligned} R_{\mathrm{ps}}(t;\boldsymbol{\eta}) &= P\left[\bigcup_{r=1}^{M}\bigcap_{j=1}^{n_r}\{T_{rj} > t\}\right] \\ &= \sum_{\emptyset\neq\mathcal{D}\subseteq\mathcal{M}} (-1)^{|\mathcal{D}|+1} L_H\left(\eta_{\mathcal{D}}^{\mathrm{ps}}; t\right). \end{aligned} \tag{39}$$

Equations (33), (34), (36), and (39) provide reliability expressions for four representative system structures under the NHPBP-driven stochastic time scale. In each case, dependence among component lifetimes is incorporated through the common random stochastic time scale $H(t)$. The resulting system reliability $R(t;\boldsymbol{\eta})$ will be used as the reliability constraint in the allocation optimization model developed in the next section.

5. Reliability allocation under NHPBP

Based on the reliability expressions derived in section 4, this section formulates a reliability allocation model under the NHPBP-driven stochastic time scale. The decision variables are the component baseline failure-rate vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_K)^T$. Since component lifetimes are dependent through the common random exposure $H(t)$, the reliability constraint is evaluated using the system reliability function $R(t;\boldsymbol{\eta})$ derived under the proposed dependence framework.

5.1. Problem formulation

Consider a system consisting of K components. The objective is to determine the component baseline failure rates $\eta_1, \eta_2, \cdots, \eta_K$, such that the total system cost is minimized while the system reliability at mission time t satisfies a prescribed target R_0. Equivalently, determining η_i specifies the reliability level assigned to component i under the stochastic time scale.

For component i, a smaller η_i corresponds to a lower baseline failure rate and a higher intrinsic reliability. Improving component reliability usually requires stricter

design requirements, better materials, higher manufacturing precision, or more intensive inspection, which leads to higher cost. To describe this cost-reliability tradeoff in the numerical allocation model, the cost of component i is specified as

$$c_i(\eta_i) = \alpha_i \exp(-\beta_i \eta_i), \tag{40}$$

where $\alpha_i > 0$ is the cost scale parameter and $\beta_i > 0$ controls the cost sensitivity to the baseline failure rate. This empirical cost function provides a monotone decreasing relationship between η_i and $c_i(\eta_i)$. It is used for illustrating the proposed reliability allocation framework, rather than representing a cost model fitted from a specific braking-system dataset. In practical engineering applications, the cost function can be obtained from historical procurement costs, design improvement records, manufacturing data, inspection costs, or expert elicitation.

The total system cost is

$$C(\boldsymbol{\eta}) = \sum_{i=1}^{K} c_i(\eta_i). \tag{41}$$

Let $R(t;\boldsymbol{\eta})$ denote the system reliability at mission time t. For a given system structure, $R(t;\boldsymbol{\eta})$ is evaluated using the corresponding expression in section 4, with $L_H(q;t)$ computed by the matrix exponential representation in equation (25).

Thus, the dependence among component lifetimes induced by the common stochastic time scale is explicitly incorporated into the allocation model.

The reliability allocation problem is formulated as

$$\begin{aligned} \min_{\boldsymbol{\eta}} \quad & C(\boldsymbol{\eta}) = \sum_{i=1}^{K} \alpha_i \exp(-\beta_i \eta_i) \\ \text{s.t.} \quad & R(t;\boldsymbol{\eta}) \geqslant R_0, \\ & \eta_{i,\min} \leqslant \eta_i \leqslant \eta_{i,\max}, \end{aligned} \tag{42}$$

where R_0 is the required system reliability level, and $\eta_{i,\min}$ and $\eta_{i,\max}$ denote the feasible lower and upper bounds of the baseline failure rate of component $i (i = 1, 2, \cdots, K)$, respectively.

In equation (42), t denotes the mission time measured from the initial operating time 0 to the end of the specified mission period. Therefore, the constraint $R(t;\boldsymbol{\eta}) \geqslant R_0$ requires that the system reliability at the end of the mission interval $[0, t]$ should not be lower than the prescribed target reliability.

5.2. Problem solution

The optimization problem in equation (42) is nonlinear and constrained. The system reliability $R(t;\boldsymbol{\eta})$ involves the matrix-exponential evaluation of $L_H(q;t)$. For redundant structures, it also involves finite summations over component subsets. An explicit analytical gradient is generally unavailable. A genetic algorithm is adopted as a derivative-free solution method.

In the genetic algorithm, each chromosome represents a candidate baseline failure-rate vector $\boldsymbol{\eta}$. For each chromosome, the system reliability $R(t;\boldsymbol{\eta})$ is evaluated according to the corresponding system structure, and the total cost $C(\boldsymbol{\eta})$ is computed by equation (41). The reliability constraint is handled as a nonlinear constraint in the GA implementation. Specifically, for each chromosome $\boldsymbol{\eta}$, the objective value is given by the total cost $C(\boldsymbol{\eta})$, while the constraint violation is evaluated by

$$g(\boldsymbol{\eta}) = R_0 - R(t;\boldsymbol{\eta}).$$

A candidate solution is feasible if $g(\boldsymbol{\eta}) \leqslant 0$, namely $R(t;\boldsymbol{\eta}) \geqslant R_0$. Therefore, the GA searches for the feasible chromosome with the minimum total cost.

The overall GA-based reliability allocation procedure is illustrated in figure 1, and its main steps are summarized as follows.

1. **Parameter initialization:** Specify the system structure, mission time t, reliability target R_0, NHPBP parameters ω, γ, β, and θ, component cost parameters α_i and β_i, feasible bounds of η_i, and GA parameters.
2. **Population generation:** Generate an initial population of chromosomes $\boldsymbol{\eta}$ within the feasible interval $[\eta_{i,\min}, \eta_{i,\max}]$ for each component.
3. **Reliability evaluation:** For each chromosome, compute $R(t;\boldsymbol{\eta})$ according to the given system structure. The required values of $L_H(q;t)$ are evaluated by
$$L_H(q;t) = \mathbf{1}^T \exp[B(q)\Lambda(t)]\,\mathbf{e}_0, \tag{43}$$
where the truncation dimension of $B(q)$ is selected by checking that the retained probability mass of the shock-counting process under $q = 0$ is close to one.
4. **Objective and constraint evaluation:** Compute the total cost $C(\boldsymbol{\eta})$ and the reliability constraint value $g(\boldsymbol{\eta}) = R_0 - R(t;\boldsymbol{\eta})$.
5. **Selection, crossover, and mutation:** Select parent chromosomes according to their objective and constraint performance, and generate offspring through crossover and mutation. If necessary, offspring are adjusted to ensure that all decision variables remain within their feasible bounds.
6. **Termination:** Repeat the above operations until the maximum number of generations is reached or the convergence tolerance is satisfied. The best feasible chromosome with the minimum total cost is taken as the allocation solution.

The obtained solution is denoted by

$$\boldsymbol{\eta}^* = (\eta_1^*, \eta_2^*, \cdots, \eta_K^*)^T.$$

The corresponding cost and system reliability are

$$C^* = C(\boldsymbol{\eta}^*), \quad R^* = R(t;\boldsymbol{\eta}^*).$$

A feasible allocation scheme should satisfy $R^* \geqslant R_0$. If the final solution does not satisfy this condition, the allocation problem is regarded as infeasible under the prescribed bounds and target reliability. The proposed procedure is structure-adaptive because only $R(t;\boldsymbol{\eta})$ needs to be specified according to the system configuration. Therefore, the same optimization framework can be applied to series, parallel, series-parallel, and parallel-series systems under the NHPBP-driven dynamic environment.

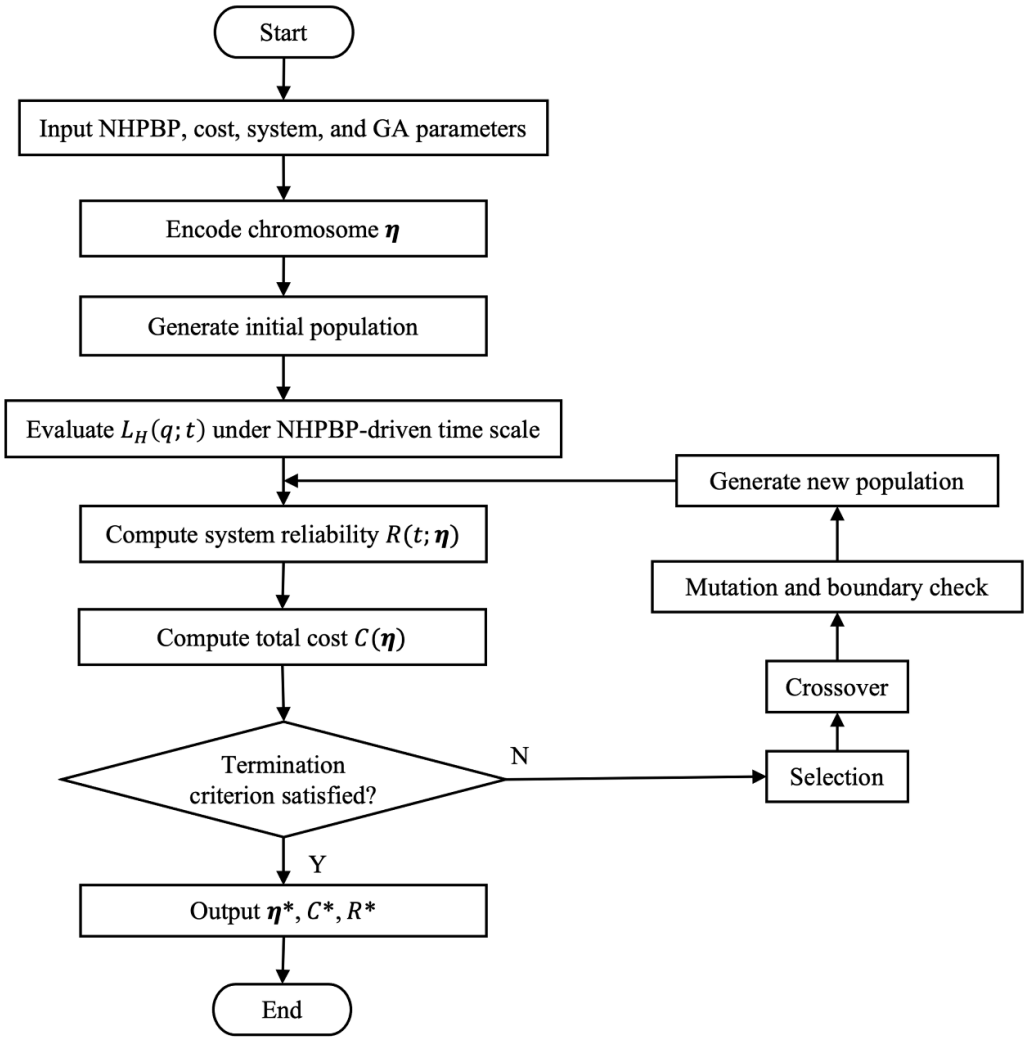


Figure 1. Flowchart of reliability allocation optimization based on genetic algorithm.

6. Numerical experiments

In this section, numerical experiments are conducted to illustrate the proposed NHPBP-driven reliability analysis and reliability allocation framework. Taking an automotive braking system as an illustrative case, we first compare reliability evaluation results under different dynamic-environment models, then investigate reliability allocation under model misspecification, and finally analyze the sensitivity of the two NHPBP parameters ω and γ.

6.1. Model application

In an automotive braking system, the braking function is regarded as failed if either the front or rear brake pad cannot provide the required braking performance. Therefore, the braking system is simplified as a two-component series system from the functional reliability perspective. During braking, the front and rear brake pads are exposed to common driving conditions, road surfaces, braking frequency, load levels, and thermal-mechanical environments. These shared dynamic factors may induce dependent wear processes and correlated lifetimes. Hong *et al* [10] collected brake pad lifespan data from 50 *t*axis put into service in 2013, which indicated positive lifetime dependence between front and rear brake pads. Therefore, the common dynamic environment and the induced component dependence should be considered in reliability analysis and reliability allocation.

Let component 1 and component 2 denote the front and rear brake pads, respectively. The mission time is set as $t = 2$ years, and the system reliability requirement is $R_0 = 0.90$. The feasible design interval of each baseline failure rate is set as $\eta_i \in [0.01, 0.5]$.

The baseline shock intensity is specified as

$$\lambda(t) = 0.5t^{0.8},$$

with cumulative baseline intensity

$$\Lambda(t) = \int_0^t 0.5x^{0.8}\mathrm{d}x = \frac{0.5}{1.8}t^{1.8}.$$

This corresponds to the Weibull-type baseline intensity in section 2 with shape parameter $\beta = 1.8$ and scale parameter satisfying $\beta/\theta^{\beta} = 0.5$.

Table 2. Main GA parameter settings.

Parameter	Value	Meaning
Mission time t	2 years	End time of the specified mission interval.
Target reliability R_0	0.90	Minimum required system reliability at mission time t.
Feasible interval of η_i	$[0.01, 0.50]$	Lower and upper bounds of the component baseline failure rate.
Population size	100	Number of candidate solutions in each generation.
Maximum generations	200	Maximum number of evolutionary iterations.
Crossover fraction	0.80	Proportion of offspring generated by crossover.
Function tolerance	10^{-10}	Stopping tolerance for objective-function convergence.
Constraint tolerance	10^{-8}	Feasibility tolerance for nonlinear constraints.
Random seed	2026	Seed used to ensure reproducibility of GA results.
Truncation dimension	90	Finite dimension used to approximate the matrix exponential.
Mutation operator	Adaptive feasible mutation	Mutation scheme used to generate feasible offspring.

According to the NHPBP model developed in section 2, the state-dependent shock intensity is

$$\lambda_k(t) = (1+\omega k)\,\lambda(t).$$

The aging-rate amplification factor is

$$b_k = 1+\gamma k.$$

Unless otherwise specified, the reference dynamic environment is set as

$$\omega = 0.3, \quad \gamma = 0.8.$$

Here, ω reflects the influence of historical shocks on the occurrence rate of future shocks, while γ reflects the influence of historical shocks on the aging rate of components. When $\omega = 0$ and $\gamma = 0$, the model reduces to the NHPP baseline without cumulative shock amplification. Positive values of these two parameters indicate that accumulated shocks may increase future shock occurrence and accelerate component aging. In the numerical example, $\omega = 0.3$ and $\gamma = 0.8$ are used as a reference setting to illustrate the proposed method, rather than as estimates from a specific braking-system dataset.

The sensitivity analysis further considers $\omega \in \{0, 0.1, 0.3, 0.5\}$ and $\gamma \in \{0, 0.3, 0.6, 0.9\}$, covering environments from the NHPP baseline to stronger cumulative amplification. In practical engineering applications, ω and γ are nonnegative parameters, and their magnitudes can be estimated or calibrated from field data such as shock-count records, degradation histories, lifetime data, or maintenance records. Maximum likelihood estimation, least-squares fitting, or Bayesian updating can be used when sufficient field data are available.

According to equation (33), the reliability of the two-component series system is

$$R_s(t;\eta_1,\eta_2) = L_H(\eta_1+\eta_2;t),$$

where

$$L_H(q;t) = \mathbf{1}^T \exp\left[B(q)\,\Lambda(t)\right]\mathbf{e}_0.$$

In numerical computation, the truncation dimension of $B(q)$ is selected by checking that the retained probability mass of the shock-counting process under $q = 0$ is close to one.

The component cost function is set as

$$c_i(\eta_i) = \alpha_i \exp(-\beta_i \eta_i),$$

where

$$\alpha_1 = \alpha_2 = 1000, \quad \beta_1 = \beta_2 = 20.$$

The main parameters used in the GA are listed in table 2. In the implementation, the decision variables are η_i, the reliability constraint is $R_{\mathrm{sys}}(t) \geqslant R_0$, and $L_H(q;t)$ is evaluated using MATLAB `expm`. Following the truncation criterion described in section 3, the truncation dimension is set to 90 in the numerical experiments.

6.2. Reliability evaluation under different dynamic environment models

To show the roles of ω and γ, four benchmark models are compared:

Model A: $\omega = 0,\ \gamma = 0$, NHPP baseline model,
Model B: $\omega = 0.3,\ \gamma = 0$, shock-rate amplification only,
Model C: $\omega = 0,\ \gamma = 0.8$, aging-rate amplification only,
Model D: $\omega = 0.3,\ \gamma = 0.8$, full NHPBP-driven model.

For illustration, the baseline failure rates are fixed as

$$\eta_1 = \eta_2 = 0.04.$$

The reliability curves of the two-component series system under the fixed baseline failure rates are shown in figure 2.

As shown in figure 2, the NHPP baseline model gives the highest reliability curve because it ignores cumulative shock amplification. At $t = 2$ years and $\eta_1 = \eta_2 = 0.04$, the reliability under Model A is 0.9255. Model B gives the same value because $\gamma = 0$ implies $b_k = 1$ for all k, and thus $H(t) = \Lambda(t)$. Therefore, although $\omega > 0$ changes the shock-counting process, it does not affect component aging when $\gamma = 0$.

By contrast, Model C reduces the system reliability to 0.8988, showing that aging-rate amplification directly

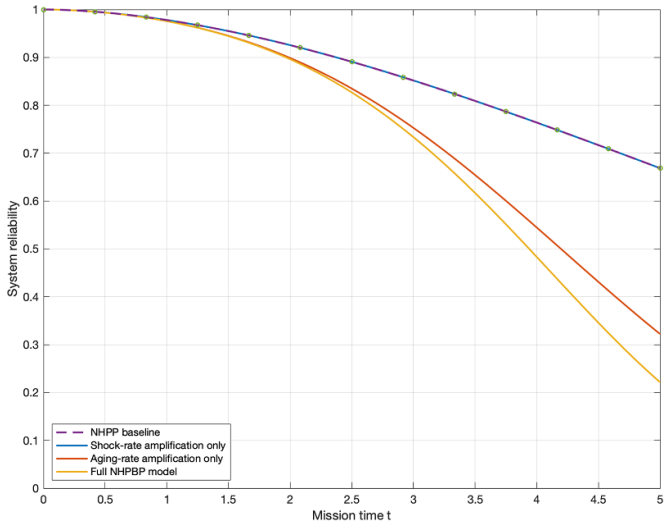


Figure 2. Reliability curves under different dynamic-environment models.

Table 3. Reliability allocation results under NHPBP and NHPP models.

		Case 1 Proposed NHPBP	Case 2 NHPP-based
η_i	Component 1	0.0385	0.0545
	Component 2	0.0384	0.0545
Cost	Component 1	463.09	336.40
	Component 2	464.20	336.54
	Total cost	927.30	672.94
Reliability used in optimization		0.9000	0.9000
Reliability under reference NHPBP		0.9000	0.8617

increases the accumulated operational time. Model D further reduces the reliability to 0.8962 because both future shock occurrence and component aging are amplified by historical shocks. These comparisons show that the proposed NHPBP-driven stochastic time scale can describe accelerated degradation caused by cumulative environmental shocks.

6.3. Comparison of different models

To illustrate the influence of model selection on reliability allocation, two cases are compared. In Case 1, the allocation model uses the proposed NHPBP-driven reliability function with $\omega = 0.3$ and $\gamma = 0.8$. In Case 2, the allocation model is based on the conventional NHPP assumption with $\omega = 0$ and $\gamma = 0$. After allocation, both schemes are re-evaluated under the reference NHPBP environment with $\omega = 0.3$ and $\gamma = 0.8$. The results are reported in table 3.

As shown in table 3, the NHPP-based allocation satisfies the reliability target under the simplified NHPP model. However, when the same allocation is re-evaluated under the reference NHPBP environment, the system reliability decreases to 0.8617, which is lower than the target value $R_0 =$ 0.90. This indicates that ignoring cumulative shock amplification may overestimate system reliability and produce an infeasible allocation scheme.

By contrast, the proposed NHPBP-based allocation incorporates both the shock occurrence amplification parameter ω and the aging-rate amplification parameter γ. Although it requires a higher total cost, it satisfies the reliability requirement under the reference dynamic environment. Thus, the advantage of the proposed model lies not in obtaining a lower nominal cost, but in providing a feasible allocation scheme under cumulative shock amplification.

6.4. Sensitivity analysis

The proposed model contains two key environmental parameters. The parameter ω controls the state-dependent amplification of future shock occurrence, while γ controls the acceleration of component aging after shocks have accumulated. To investigate their effects, ω and γ are varied as

$$\omega \in \{0, 0.1, 0.3, 0.5\}, \quad \gamma \in \{0, 0.3, 0.6, 0.9\}.$$

For each parameter combination, the reliability allocation problem is solved under the corresponding NHPBP-driven model, and the resulting total cost is recorded. In addition, the allocation obtained under the NHPP assumption is re-evaluated under each NHPBP environment. The results are shown in figure 3.

Figure 3(a) shows that the total cost generally increases as ω and γ increase. When $\gamma = 0$, the stochastic time scale reduces to $H(t) = \Lambda(t)$, and the allocation result is not affected by ω. When $\gamma > 0$, a larger ω increases the probability of entering higher shock states, which further strengthens aging acceleration and increases the required reliability investment.

Figure 3(b) shows the consequence of model misspecification. The allocation obtained under the NHPP assumption satisfies the target reliability only when the NHPBP environment is close to the NHPP baseline. As ω and γ increase, the re-evaluated reliability decreases significantly and may fall below the target value. This confirms that the NHPP-based model may underestimate the required component reliability in dynamic environments with cumulative shock amplification.

Figure 3(c) shows that a larger γ leads to faster reliability degradation because the same shock history produces a larger accumulated operational time. Figure 3(d) shows that increasing ω also reduces reliability when $\gamma > 0$, since the system is more likely to enter higher shock states and accumulate stronger aging exposure. These results demonstrate that the proposed NHPBP-driven stochastic time scale provides a flexible and physically interpretable framework for reliability analysis and allocation under dynamic operating environments. They also show that the allocation model should be consistent with the dynamic-environment assumptions used for reliability evaluation.

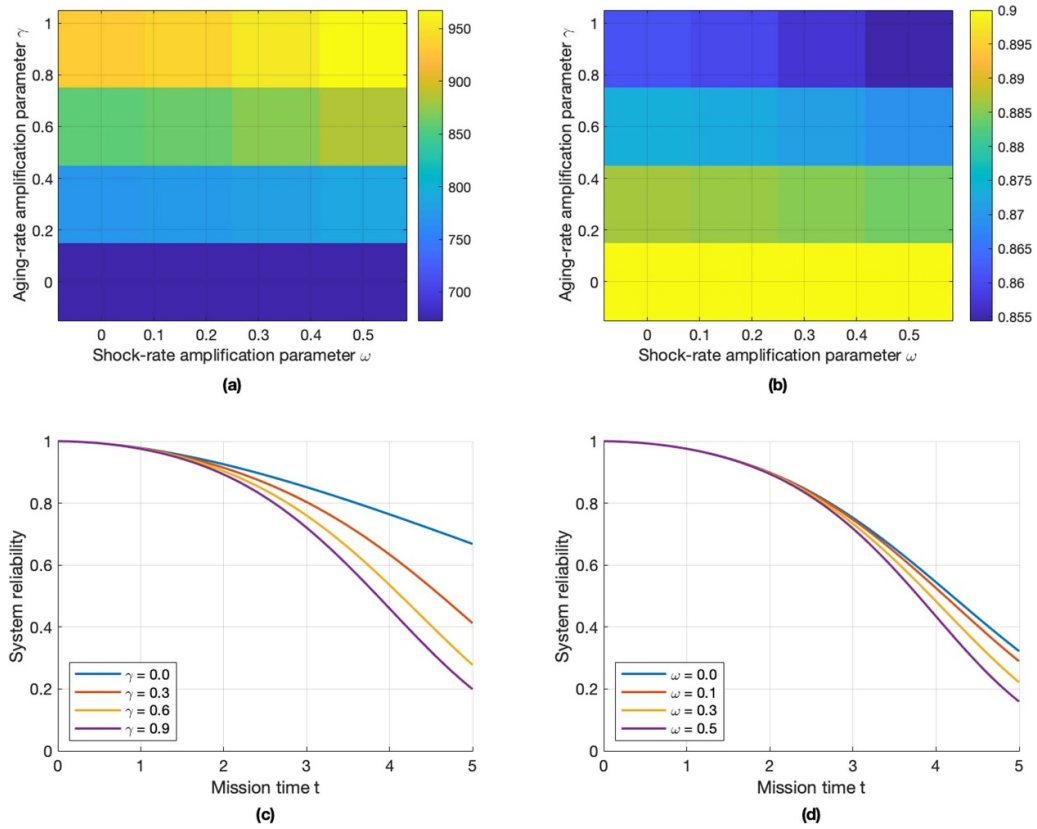


Figure 3. Sensitivity analysis of NHPBP parameters ω and γ. (a) Total cost of the NHPBP-based allocation. (b) Reliability of the NHPP-based allocation re-evaluated under NHPBP environments. (c) Reliability curves under different values of γ with $\omega = 0.3$. (d) Reliability curves under different values of ω with $\gamma = 0.8$.

The numerical results above provide several practical implications for reliability allocation under dynamic environments. The NHPP-based allocation achieves the target reliability under the simplified model, but its reliability decreases below the target level when it is re-evaluated under the reference NHPBP environment. This result indicates that ignoring cumulative shock amplification may lead to an insufficient reliability investment. For reliability engineers, the allocation model should be selected according to the actual operating environment, rather than only according to the nominal allocation cost.

Although the numerical example is a two-component series system, the proposed framework is not limited to this structure. The reliability expressions derived in section 4 cover series, parallel, series-parallel, and parallel-series systems. The GA-based allocation procedure uses $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_K)^T$ as the chromosome, so the same procedure can be extended to systems with more components by changing the system reliability function $R(t; \boldsymbol{\eta})$. For larger systems, the main additional computational cost comes from the evaluation of $L_H(q;t)$ and the structure-dependent reliability summations.

7. Conclusion

This paper studied the reliability analysis and reliability allocation of multicomponent systems operating under dynamic environments. An NHPBP-driven stochastic time-scale model was developed to describe cumulative environmental shocks. In the proposed model, $a_k = 1 + \omega k$ characterizes the effect of accumulated shocks on future shock occurrence, while $b_k = 1 + \gamma k$ characterizes the effect of accumulated shocks on component aging. Based on this stochastic time scale, the component survival function was expressed through the Laplace transform $L_H(q;t)$, and a matrix exponential representation was derived for reliability evaluation. The joint survival function and positive quadrant dependence of component lifetimes were also established. Furthermore, reliability expressions were derived for series, parallel, series-parallel, and parallel-series systems under the common stochastic time scale.

A cost-oriented reliability allocation model was then formulated under a system-level reliability constraint and solved by a GA-based procedure. Numerical experiments on an automotive braking system showed that the conventional NHPP-based model may overestimate system reliability when cumulative shock amplification exists, leading to an infeasible allocation scheme under the reference dynamic environment. In contrast, the proposed NHPBP-based allocation model explicitly incorporates shock-rate amplification and aging-rate amplification, and can provide feasible allocation results under dynamic environments. Sensitivity analysis further showed that ω and γ affect system reliability and allocation cost through different but coupled mechanisms.

The proposed framework provides a tractable reliability analysis and allocation method for multicomponent systems subject to cumulative environmental shocks. Future research will focus on estimating model parameters from field data, incorporating imperfect maintenance and load-sharing effects, and extending the framework to systems with multiple failure modes and more complex dependence mechanisms.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant numbers 72401132, China Postdoctoral Science Foundation under Grant number 2022M721596, and the Natural Science Foundation of Jiangsu Province under Grant number BK20230870.

Appendix A. State probabilities and non-explosiveness of the NHPBP

Let

$$p_k(t) = P\{N(t) = k\}, \quad k = 0,1,2,\cdots.$$

For the NHPBP with $\lambda_k(t) = a_k\lambda(t)$, the state probabilities satisfy

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -a_0\lambda(t)p_0(t),$$

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = a_{k-1}\lambda(t)p_{k-1}(t) - a_k\lambda(t)p_k(t),$$

with initial conditions

$$p_0(0) = 1, \quad p_k(0) = 0, \quad k \geqslant 1.$$

To simplify the derivation, introduce the operational time

$$u = \Lambda(t) = \int_0^t \lambda(x)\,\mathrm{d}x,$$

and denote $q_k(u) = p_k(t)$. Then

$$\frac{\mathrm{d}q_0(u)}{\mathrm{d}u} = -a_0 q_0(u),$$

$$\frac{\mathrm{d}q_k(u)}{\mathrm{d}u} = a_{k-1}q_{k-1}(u) - a_k q_k(u).$$

Thus,

$$q_0(u) = \exp(-a_0 u).$$

For $k \geqslant 1$, multiplying both sides by $\exp(a_k u)$ gives the recursive integral solution

$$q_k(u) = \int_0^u a_{k-1}q_{k-1}(v)\exp[-a_k(u-v)]\,\mathrm{d}v.$$

Equivalently,

$$p_k(t) = \int_0^{\Lambda(t)} a_{k-1}q_{k-1}(v)\exp[-a_k(\Lambda(t)-v)]\,\mathrm{d}v.$$

Under the linear amplification form

$$a_k = 1 + \omega k, \quad \omega \geqslant 0,$$

we have $a_0 = 1$, and therefore

$$p_0(t) = \exp[-\Lambda(t)].$$

When $\omega > 0$, the state probabilities $\{p_k(t), k = 0,1,2,\cdots\}$ can be written as

$$p_k(t) = \exp[-\Lambda(t)]\frac{(1/\omega)_k}{k!}[1-\exp[-\omega\Lambda(t)]]^k,$$

where

$$(1/\omega)_k = \frac{1}{\omega}\left(\frac{1}{\omega}+1\right)\cdots\left(\frac{1}{\omega}+k-1\right)$$

is the rising factorial, with $(1/\omega)_0 = 1$. This expression provides a valid probability mass function for the shock-counting process. In the special case $\omega = 0$, the NHPBP reduces to an NHPP, and the state probabilities become

$$p_k(t) = \exp[-\Lambda(t)]\frac{[\Lambda(t)]^k}{k!}.$$

Finally, the linear amplification form is non-explosive. For $\omega = 0$, this is immediate since $a_k = 1$. For $\omega > 0$,

$$\sum_{k=0}^{\infty}\frac{1}{a_k} = \sum_{k=0}^{\infty}\frac{1}{1+\omega k} = \infty,$$

which satisfies the standard non-explosiveness condition for pure birth processes. Hence, for any finite t with $\Lambda(t) < \infty$, the NHPBP does not produce infinitely many shocks in finite time.

Appendix B. Derivation of the Laplace transform of the stochastic time scale

The weighted state probability $\{m_k(t;q), k = 0,1,2,\cdots\}$ is defined as

$$m_k(t;q) = E\left[\exp[-qH(t)]\mathbf{1}_{\{N(t)=k\}}\right].$$

Then the Laplace transform of $H(t)$ can be written as

$$L_H(q;t) = E\{\exp[-qH(t)]\} = \sum_{k=0}^{\infty} m_k(t;q).$$

When $N(t) = k$, the NHPBP leaves state k at rate $a_k\lambda(t)$. Meanwhile, since

$$H(t) = \int_0^t b_{N(s)}\,\mathrm{d}\Lambda(s),$$

the stochastic time scale increases locally at rate $b_k\lambda(t)$. Therefore, over a small interval $(t, t+\Delta t]$, the weighted state probability loses mass due to both the transition from state k to state $k+1$ and the exponential factor $\exp[-qH(t)]$. This gives

$$\frac{\mathrm{d}m_0(t;q)}{\mathrm{d}t} = -(a_0+qb_0)\lambda(t)m_0(t;q),$$

and, for $k \geqslant 1$,

$$\frac{\mathrm{d}\mathbf{m}(t;q)}{\mathrm{d}t} = a_{k-1}\lambda(t)m_{k-1}(t;q) - (a_k+qb_k)\lambda(t)m_k(t;q).$$

The initial conditions are

$$m_0(0;q) = 1, \quad m_k(0;q) = 0.$$

Introducing the operational time

$$u = \Lambda(t) = \int_0^t \lambda(x)\,\mathrm{d}x,$$

since $\mathrm{d}u/\mathrm{d}t = \lambda(t)$, the above equations become

$$\frac{\mathrm{d}m_0(u;q)}{\mathrm{d}u} = -(a_0+qb_0)m_0(u;q),$$

$$\frac{\mathrm{d}m_k(u;q)}{\mathrm{d}u} = a_{k-1}m_{k-1}(u;q) - (a_k+qb_k)m_k(u;q).$$

Let $\mathbf{m}(u;q) = (m_0(u;q), m_1(u;q), m_2(u;q), \cdots)^T$, and $\mathbf{e}_0 = (1,0,0,\cdots)^T$. Then the system can be written as

$$\frac{\mathrm{d}\mathbf{m}(u;q)}{\mathrm{d}u} = B(q)\mathbf{m}(u;q),$$

$$\mathbf{m}(0;q) = \mathbf{e}_0,$$

where $B(q)$ is a lower bidiagonal matrix with nonzero entries

$$B(q)_{k,k} = -(a_k+qb_k), \quad k = 0,1,2,\cdots,$$

$$B(q)_{k,k-1} = a_{k-1}, \quad k \geqslant 1.$$

Thus,

$$\mathbf{m}(u;q) = \exp[B(q)u]\,\mathbf{e}_0.$$

Substituting $u = \Lambda(t)$ and summing over all states, we obtain

$$L_H(q;t) = \mathbf{1}^T \exp[B(q)\Lambda(t)]\,\mathbf{e}_0,$$

where

$$\mathbf{1} = (1,1,1,\cdots)^T.$$

This completes the derivation.

Appendix C. Joint survival function and PQD proof

Consider two components s and m with lifetimes T_s and T_m. For $0 \leqslant t_s \leqslant t_m$, conditional on the environmental path, we have

$$P\{T_s > t_s, T_m > t_m \mid H\} = \exp[-\eta_s H(t_s)]\exp[-\eta_m H(t_m)].$$

Since

$$H(t_m) = H(t_s) + [H(t_m) - H(t_s)],$$

it follows that

$$P\{T_s > t_s, T_m > t_m \mid H\} = \exp[-(\eta_s+\eta_m)H(t_s)]\exp[-\eta_m(H(t_m)-H(t_s))].$$

The first factor corresponds to the cumulative exposure in $[0, t_s]$, where both components must survive, while the second factor corresponds to the cumulative exposure in $(t_s, t_m]$, where only component m must survive.

Using the Markov property of the NHPBP and the matrix representation of the stochastic time scale, for $0 \leqslant t_s \leqslant t_m$, we obtain

$$\begin{aligned} S_{s,m}(t_s,t_m) &= P\{T_s > t_s, T_m > t_m\} \\ &= \mathbf{1}^T \exp\{B(\eta_m)[\Lambda(t_m) - \Lambda(t_s)]\} \\ &\quad \times \exp\{B(\eta_s+\eta_m)\Lambda(t_s)\}\,\mathbf{e}_0. \end{aligned}$$

Similarly, for $0 \leqslant t_m < t_s$,

$$\begin{aligned} P\{T_s > t_s, T_m > t_m \mid H\} &= \exp[-(\eta_s+\eta_m)H(t_m)] \\ &\quad \times \exp[-\eta_s(H(t_s) - H(t_m))], \end{aligned}$$

and therefore

$$\begin{aligned} S_{s,m}(t_s,t_m) &= \mathbf{1}^T \exp\{B(\eta_s)[\Lambda(t_s) - \Lambda(t_m)]\} \\ &\quad \times \exp\{B(\eta_s+\eta_m)\Lambda(t_m)\}\,\mathbf{e}_0. \end{aligned}$$

When $t_s = t_m = t$, both expressions reduce to

$$S_{s,m}(t,t) = L_H(\eta_s+\eta_m;t).$$

Next, we prove the positive quadrant dependence of T_s and T_m. Let

$$X = \exp[-\eta_s H(t_s)],$$

$$Y = \exp[-\eta_m H(t_m)].$$

Then

$$P\{T_s > t_s, T_m > t_m\} = E(XY),$$

$$P\{T_s > t_s\} = E(X),$$

$$P\{T_m > t_m\} = E(Y).$$

In the operational time $u = \Lambda(t)$, the NHPBP can be represented by independent interarrival times $Z_k \sim \mathrm{Exp}(a_k)$, where $a_k = 1 + \omega k$. The exposure process can be written as

$$\widetilde{H}(u) = \int_0^u b_{N(v)}\,\mathrm{d}v,$$

with $b_k = 1 + \gamma k$. Since b_k is nondecreasing in k, increasing any interarrival time Z_k delays subsequent shocks and cannot increase $\widetilde{H}(u)$. Therefore, $H(t_s)$ and $H(t_m)$ are nonincreasing functions of the independent interarrival-time sequence $\{Z_k\}_{k\geqslant 0}$. Consequently, X and Y are coordinatewise nondecreasing functions of $\{Z_k\}_{k\geqslant 0}$.

By the association property of independent random variables, two coordinatewise monotone functions of independent random variables have nonnegative covariance. Hence,

$$\mathrm{Cov}\,(X, Y) \geqslant 0,$$

which gives

$$E(XY) \geqslant E(X)E(Y).$$

Thus,

$$P\{T_s > t_s, T_m > t_m\} \geqslant P\{T_s > t_s\}P\{T_m > t_m\},$$

for any $t_s, t_m \geqslant 0$. Therefore, T_s and T_m are positively quadrant dependent.

ORCID iDs

Zhou Fang 0009-0007-5091-6366
Wenjie Dong 0000-0002-8591-1756
Yingsai Cao 0000-0003-1487-4553

References

[1] Luo Y, Zhao X, Liu B and He S 2024 Condition-based maintenance policy for systems under dynamic environment *Reliab. Eng. Syst. Saf.* **246** 110072
[2] Hu J and Sun Q 2025 A dynamic inspection and replacement policy for systems subject to degradation and periodic shocks *Reliab. Eng. Syst. Saf.* **266** 111765
[3] Wang Y, Liu F, Xi F, Wei B, Duan D, Cai Z and Si S 2025 Data driven comprehensive performance evaluation of aeroengines: a network dynamic approach *Engineering* **46** 292–305
[4] Jia H, Xing L, Ding Y, Li Y and Liu D 2023 Reliability analysis of dynamic load-sharing systems with constrained and changing component performances *IEEE Trans. Syst. Man Cybern.: Syst.* **53** 5897–909
[5] Luo C, Shen L and Xu A 2022 Modelling and estimation of system reliability under dynamic operating environments and lifetime ordering constraints *Reliab. Eng. Syst. Saf.* **218** 108136
[6] Dong W, Liu S, Bae S J and Cao Y 2021 Reliability modelling for multi-component systems subject to stochastic deterioration and generalized cumulative shock damages *Reliab. Eng. Syst. Saf.* **205** 107260
[7] Gu Y-K, Fan C-J, Liang L-Q and Zhang J 2022 Reliability calculation method based on the Copula function for mechanical systems with dependent failure *Ann. Oper. Res.* **311** 99–116
[8] Sung C-K, Sheu S-H, Hsu T-S and Chen Y-C 2013 Extended optimal replacement policy for a two-unit system with failure rate interaction and external shocks *Int. J. Syst. Sci.* **44** 877–88
[9] Tan L, Wei F, Ma X, Peng R, Xiao H and Yang Li 2025 Systemic condition-based maintenance optimization under inspection uncertainties: a customized multiagent reinforcement learning approach *IEEE Trans. Reliab.* **74** 5848–62
[10] Hong L, Zhai Q, Wang X and Ye Z-S 2018 System reliability evaluation under dynamic operating conditions *IEEE Trans. Reliab.* **68** 800–9
[11] Wu Di, Gao K, Peng R, Sun J, Yang Li and Wei F 2025 Optimal preventive maintenance policy considering technical-innovation-empowered renewals *Proc. Inst. Mech. Eng. O* **8** 1748006X251412931
[12] Yang Li, Ma Y, Wei F and Qiu Q 2025 Hybrid risk control of mission-critical systems under degradation uncertainties: integrated maintenance-termination optimization *Expert Syst. Appl.* **300** 130444
[13] Sun Q, Hu J and Ye Z-S 2025 Optimal abort policy for mission-critical systems under imperfect condition monitoring *Oper. Res.* **73** 2396–416
[14] Joe H 1997 *Multivariate Models and Multivariate Dependence Concepts* (Oxford University Press)
[15] Peng W, Li Y-F, Yang Y-J, Zhu S-P and Huang H-Z 2016 Bivariate analysis of incomplete degradation observations based on inverse Gaussian processes and copulas *IEEE Trans. Reliab.* **65** 624–39
[16] Liu P and Wang G 2024 Generalized Pólya-process-based reliability analysis for systems working under dynamic environment *IEEE Trans. Reliab.* **74** 2146–56
[17] Nelson W 1980 Accelerated life testing-step-stress models and data analyses *IEEE Trans. Reliab.* **29** 103–8
[18] Duchesne T and Lawless J 2000 Alternative time scales and failure time models *Lifetime Data Anal.* **6** 157–79
[19] Whitmore G A and Schenkelberg F 1997 Modelling accelerated degradation data using Wiener diffusion with a time scale transformation *Lifetime Data Anal.* **3** 27–45
[20] Xu A, Zhou S and Tang Y 2019 A unified model for system reliability evaluation under dynamic operating conditions *IEEE Trans. Reliab.* **70** 65–72
[21] Jiang T, Liu Y and Ye Z-S 2023 A stochastic time scale based framework for system reliability under a Markovian dynamic environment *Nav. Res. Logist. (NRL)* **70** 320–39
[22] Elegbede A O C, Chu C, Adjallah K H and Yalaoui F 2003 Reliability allocation through cost minimization *IEEE Trans. Reliab.* **52** 106–11
[23] Zhang D, Hao Z, Liang Y, Wang F, Liu W and Han X 2024 An efficient system reliability analysis method based on evidence theory with parameter correlations *IEEE Trans. Reliab.* **74** 2200–13
[24] Wang D, Liu M, Yang H and Si S 2024 A novel importance measure considering multi-constraints for rap optimization of 1-out-of-n subsystems with mixed redundancy strategy *Reliab. Eng. Syst. Saf.* **252** 110441
[25] Ye Z, Cai Z, Yang H, Si S and Zhou F 2023 Joint optimization of maintenance and quality inspection for manufacturing networks based on deep reinforcement learning *Reliab. Eng. Syst. Saf.* **236** 109290
[26] Coit D W and Smith A E 2005 Reliability optimization of series-parallel systems using a genetic algorithm *IEEE Trans. Reliab.* **45** 254–60
[27] Sheu S-H, Chang C-C, Zhang Z G and Chien Y-H 2012 A note on replacement policy for a system subject to

non-homogeneous pure birth shocks *Eur. J. Oper. Res.* **216** 503–8
[28] Esary J D and Marshall A W 1973 Shock models and wear processes *Ann. Probab.* **1** 627–49
[29] hameed M S A and Proschan F 1973 Nonstationary shock models *Stoch. Process. Appl.* **1** 383–404
[30] Hameed M S A and Proschan F 1975 Shock models with underlying birth process *J. Appl. Probab.* **12** 18–28
[31] Sheu S-H, Chen Y-L, Chang C-C and Zhang Z G 2012 A block replacement policy for systems subject to non-homogeneous pure birth shocks *IEEE Trans. Reliab.* **61** 741–8
[32] Chien Y-H 2019 The optimal preventive-maintenance policy for a NHPBP repairable system under free-repair warranty *Reliab. Eng. Syst. Saf.* **188** 444–53
[33] Sheu S-H, Chen Y-L, Chang C-C and Zhang Z G 2013 Optimal number of repairs before replacement for a system subject to shocks of a non-homogeneous pure birth process *IEEE Trans. Reliab.* **62** 73–81
[34] Chien Y-H, Zhang Z G, Su S and Sheu S-H 2021 Optimal periodic preventive maintenance policy for a system subject to failures/repairs which follow the non-homogeneous pure birth process *Qual. Technol. Quant. Manage.* **18** 285–97
[35] Shaked M, Spizzichino F and Suter F 2002 Nonhomogeneous birth processes and l^{∞}-spherical densities, with applications in reliability theory *Probab. Eng. Inf. Sci.* **16** 271–88



Zhou Fang received her bachelor's degree in Management, majoring in Quality Management Engineering, from China Jiliang University, China, in 2024. She is currently pursuing a master's degree in Industrial Engineering and Management at Nanjing University of Aeronautics and Astronautics, China. Her current research interests include quality and reliability, maintenance strategy optimization.



Wenjie Dong received his PhD degree in Management Science and Engineering from Nanjing University of Aeronautics and Astronautics, China, in 2021. He is currently an Associate Professor at the College of Economics and Management, Nanjing University of Aeronautics and Astronautics. His research interests include industrial engineering, reliability management, and maintenance strategy optimization. He now serves as a member of the Youth Editorial Board of *Journal of Reliability Science and Engineering* (JRSE), *Industrial Engineering, and Engineering.*



Yingsai Cao received his PhD degree in Management Science and Engineering from Nanjing University of Aeronautics and Astronautics, China, in 2019. He is currently an Associate Professor at the School of Management, Jiangsu University. His current research interests focus on system reliability analysis and optimization.